MAHARAM EXTENSION AND STATIONARY STABLE PROCESSES

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Abstract. We give a second look at stationary stable processes by interpreting the self-similar property at the level of the Lévy measure as characteristic of a Maharam system. This allows us to derive structural results and their ergodic consequences.

1. Introduction

In a fundamental paper [9], Rosiński revealed the hidden structure of stationary symmetric α-stable ($S\alpha S$) processes. Namely, he proved that, through what is called, following Hardin [5], a *minimal spectral representation*, such a process is driven by a non-singular dynamical system.

Such a result was proved to classify those processes according to their ergodic properties such as various kinds of mixing. In [13], we used a different approach as we considered the whole family of stationary infinitely divisible processes without Gaussian part (called *IDp processes*). The key tool there was the Lévy measure system of the process, which was measure-preserving and not just merely non-singular. As of today, in the stable case, it remained unclear what the connection between the Lévy measure and the non-singular system was. This is the purpose of this paper to fill the gap and go beyond both approaches.

Indeed, we will prove that Lévy measure systems of α-stable processes have the form of a so-called Maharam system. This observation has some interesting consequences as it allows us to derive very quickly minimal spectral representations in the $S\alpha S$ case, to reinforce factorization results, and to refine ergodic classification.

Let us explain very loosely the mathematical features of stable distributions we will be using. Observe that stable distributions are characterized by a self-similar property which is obvious when observing the corresponding Lévy process:

If X_t is an α-stable Lévy process, then $b^{-\frac{1}{\alpha}}X_{bt}$ has the same distribution.

However, if not obvious or useful, this property is also present for any α-stable object but takes another form. The common feature is to be found in the Lévy measure:

Loosely speaking, if $\{X_t\}_{t\in S}$ is an α-stable process indexed by a set S, then for any positive number c, the image of the Lévy measure Q by the map $R_c := \{x_t\}_{t\in S} \mapsto \{cx_t\}_{t\in S}$ is $c^{-\alpha}Q$.

This property of the Lévy measure is characteristic of α-stable processes and can be translated into an ergodic theoretic statement:

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The measurable non-singular flow $\{R_c\}_{c\in\mathbb{R}_+}$ is dissipative and the multiplicative coefficient $c^{-\alpha}$ has an outstanding importance in that matter, since it reveals the structure of a Maharam transformation. The importance is even greater when there is more invariance involved (stationary α-stable processes, etc...), as in the present paper.

The paper is organised as follows. In Section 2 we recall what a spectral representation is and in Section 3, we give the necessary background in non-singular ergodic theory. Maharam systems are introduced in Section 4 and the link with Lévy measures of stable processes, together with spectral representations is exposed in Section 5. Section 6 is a refinement of the structure of stable processes. We deduce from the preceding results some ergodic properties in Section 7.

2. Spectral representation

We warn the reader that we will, most of the time, omit the implicit "μ-a.e." or "modulo null sets" throughout the document.

A family of functions $\{f_t\}_{t\in T} \subset L^{\alpha}(\Omega, \mathcal{F}, \mu)$ where $(\Omega, \mathcal{F}, \mu)$ is a σ-finite Lebesgue space is said to be a spectral representation of $S\alpha S$ process $\{X_t\}_{t\in T}$ if

$$\{X_t\}_{t\in T} = \left\{\int_{\Omega} f_t(\omega)\, M(\mathrm{d}\omega)\right\}_{t\in T}$$

holds in distribution, M being an independently scattered $S\alpha S$-random measure on $(\Omega, \mathcal{F})$ with intensity measure μ.

We'll say that a spectral representation is *proper* if $\operatorname{Supp}\{f_t,\, t\in T\} = \Omega$. Of course we obtain a proper representation from a general one by removing the complement of $\operatorname{Supp}\{f_t,\, t\in T\}$.

To express that a representation contains the strict minimum to define the process, the notion of minimality has been introduced (Hardin [5]):

A spectral representation is said to be $\{f_t\}_{t\in T} \subset L^{\alpha}(\Omega, \mathcal{F}, \mu)$ *minimal* if it is proper and $\sigma\left(\frac{f_t}{f_s} 1_{\{f_s\neq 0\}},\, s,t\in T\right) = \mathcal{F}$.

Hardin proved in [5] the existence of minimal representations for $S\alpha S$ processes.

In the stationary case ($T = \mathbb{R}$ or $\mathbb{Z}$), Rosiński has explained the form of the spectral representation:

Theorem 1. *(Rosiński) Let $\{f_t\}_{t\in T} \subset L^{\alpha}(\Omega, \mathcal{F}, \mu)$ be a minimal representation of a stationary $S\alpha S$-process, then there exists nonsingular flow $\{\phi_t\}_{t\in T}$ on $(\Omega, \mathcal{F}, \mu)$ and a cocycle $\{a_t\}_{t\in T}$ for this flow with values in $\{-1,1\}$ (or in $|z| = 1$ in the complex case) such that, for each $t \in T$,*

$$f_t = a_t \left\{\frac{d\mu\circ\phi_t}{d\mu}\right\}^{\frac{1}{\alpha}} (f_0\circ\phi_t)\,.$$

3. Some terminology

A quadruplet $(\Omega, \mathcal{F}, \mu, T)$ is called a *dynamical system* or shortly a *system* if T is a *non-singular automorphism* that is a bijective bi-measurable map such that $T^*\mu \sim \mu$. If $T_*(\mu) = \mu$ then $(\Omega, \mathcal{F}, \mu, T)$ is a *measure-preserving* (abr. *m.p.*) dynamical system.

A system $(\Omega_2, \mathcal{F}_2, \mu_2, T_2)$ is said to be a *non-singular (resp. measure preserving) factor* of the system $(\Omega_1, \mathcal{F}_1, \mu_1, T_1)$ if there exists a measurable *non-singular (resp.*

measure-preserving) homomorphism between them, that is a measurable map Φ from Ω_1 to Ω_2 such that $\Phi T_1 = T_2 \Phi$ and $\Phi^* \mu_1 \sim \mu_2$ (resp. $\Phi^* \mu_1 = \mu_2$) . If Φ is invertible and bi-measurable it is called a *non-singular (resp. measure-preserving) isomorphism* and the system are said to be *non-singular (resp. measure-preserving) isomorphic.*

3.1. **Krieger types.** Consider a non-singular dynamical system $(\Omega, \mathcal{F}, \mu, T)$. A set $A \in \mathcal{F}$ such that $\mu(A) > 0$ is said to be *periodic* of period n if $T^i A$, $0 \leq i \leq n-1$, are disjoint and $T^n A = A$ and *wandering* if $T^i A$, $i \in \mathbb{Z}$ are disjoint. A set is *exhaustive* if $\cup_{k \in \mathbb{Z}} T^k A = \Omega$. A system is *conservative* if there is no wandering set and *dissipative* if there is an exhaustive wandering set.

$(\Omega, \mathcal{F}, \mu, T)$ is said to be of *Krieger type*:

- I_n if there exists an exhaustive set of period n.
- I_∞ if it is dissipative.
- II_1 if there is no periodic set and exists an equivalent finite T-invariant measure.
- II_∞ if is is conservative with an equivalent infinite T-invariant continuous measure but no absolutely continuous finite T-invariant measure.
- III if there is no absolutely continuous T-invariant measure.

4. Maharam transformation

Definition 2. A m.p. dynamical system is said to be *Maharam* if it is isomorphic to $\left(\Omega \times \mathbb{R}, \mathcal{F} \otimes \mathcal{B}, \mu \otimes \mathrm{e}^s \mathrm{d}s, \widetilde{T}\right)$ where T is a non singular automorphism of $(\Omega, \mathcal{F}, \mu)$ and $\widetilde{T}$ is defined by

$$\widetilde{T}(\omega, s) := \left(T(\omega), s - \ln \frac{\mathrm{d}T_*^{-1}\mu}{\mathrm{d}\mu}(\omega)\right).$$

Observe that the dissipative flow $\{\tau_t\}_{t \in \mathbb{R}}$ defined by $\tau_t := (\omega, s) \mapsto (\omega, s-t)$ commutes with $\widetilde{T}$.

Note that we have chosen the usual additive representation but we could (and eventually will !) use the following multiplicative representation of a Maharam system. Take $0 < \alpha < 2$, we can represent $\left(\Omega \times \mathbb{R}, \mathcal{F} \otimes \mathcal{B}, \mu \otimes \mathrm{e}^s \mathrm{d}s, \widetilde{T}\right)$ by the system $\left(\Omega \times \mathbb{R}_+^*, \mathcal{F} \otimes \mathcal{B}_+, \mu \otimes \frac{1}{s^{1+\alpha}} \mathrm{d}s, \widetilde{T}_\alpha\right)$ where $\widetilde{T}_\alpha$ is defined by:

$$\widetilde{T}_\alpha(\omega, s) := \left(T(\omega), s\left(\frac{\mathrm{d}T_*^{-1}\mu}{\mathrm{d}\mu}(\omega)\right)^{\frac{1}{\alpha}}\right).$$

The isomorphism being provided by the map $(\omega, s) \mapsto \left(\omega, (2-\alpha)^{-\frac{1}{2-\alpha}} \mathrm{e}^{(2-\alpha)s}\right)$. Observe that, under this isomorphism, $\{\tau_t\}_{t \in \mathbb{R}}$ is changed into $\left\{S_{\mathrm{e}^{\frac{t}{\alpha}}}\right\}_{t \in \mathbb{R}_+^*}$ where S_t is the multiplication by t on the second coordinate.

In [2], the authors proved the following characterization, as a straightforward application of Krengel's representation of dissipative transformations:

Theorem 3. *A system $(X, \mathcal{A}, \nu, \gamma)$ is Maharam if and only if there exists a measurable flow $\{Z_t\}_{t \in \mathbb{R}}$ commuting with γ such that $(Z_t)_* \nu = e^t \nu$. $\{Z_t\}_{t \in \mathbb{R}}$ corresponds*

to $\{\tau_t\}_{t\in\mathbb{R}}$ *under the isomorphism with the Maharam system under the additive representation.*

In the original theorem they assumed ergodicity of γ to prove that the resulting non-singular transformation T in the above representation was actually living on a non-atomic measure space $(\Omega, \mathcal{F}, \mu)$. The ergodicity assumption is therefore not necessary in the way we present this theorem.

We end this section by a very natural lemma which is part of folklore. We omit the proof.

Lemma 4. *Consider two Maharam systems* $\left(\Omega_1 \times \mathbb{R}_+^*, \mathcal{F}_1 \otimes \mathcal{B}, \mu_1 \otimes \frac{1}{s^{1+\alpha}}ds, \widetilde{T_1}\right)$ *and* $\left(\Omega_2 \times \mathbb{R}_+^*, \mathcal{F}_2 \otimes \mathcal{B}, \mu_2 \otimes \frac{1}{s^{1+\alpha}}ds, \widetilde{T_2}\right)$ *and denote by* $\{S_t\}_{t\in\mathbb{R}_+^*}$ *and* $\{Z_t\}_{t\in\mathbb{R}_+^*}$ *their respective multiplicative flows. Assume there exists a (measure-preserving) factor map (resp. isomorphism)* Φ *between the two systems such that, for all* $t \in \mathbb{R}_+^*$, $S_t\Phi = Z_t\Phi$. *Then* Φ *induces a non-singular factor map (resp. isomorphism)* ϕ *between* $(\Omega_1, \mathcal{F}_1, \mu_1, T_1)$ *and* $(\Omega_2, \mathcal{F}_2, \mu_2, T_1)$.

Remark 5. Observe also that the Maharam systems associated to $(\Omega, \mathcal{F}, \mu_1, T)$ and $(\Omega, \mathcal{F}, \mu_2, T)$ where $\mu_1 \sim \mu_2$ are isomorphic.

4.1. **Refinements of type III (see [3]).** Since the flow $\{S_t\}_{t\in\mathbb{R}}$ commutes with $\widetilde{T}$, it acts non-singularly on the space (Z, ν) of ergodic components of $\widetilde{T}$ and is called the *associated flow* of T. This flow is ergodic whenever T is ergodic and its form allows to classify ergodic type III systems:

- T is of type III_λ, $0 < \lambda < 1$ if the associated flow is the periodic flow $x \mapsto x + t \bmod (-\log\lambda)$.
- T is of type III_0 if the associated flow is free.
- T is of type III_1 if the associated flow is the trivial flow on a singleton.

In particular $\widetilde{T}$ is ergodic if and only if T is of type III_1.

5. Lévy measure as Maharam system and spectral representations

5.1. **Lévy measure of α-stable processes.** For simplicity we will only consider discrete time stationary processes.

Let us recall, following [8] (see also [13]), that the Lévy measure of stationary IDp process X of distribution $\mathbb{P}$ is the shift-invariant σ-finite measure on $\mathbb{R}^{\mathbb{Z}}$, Q, such that $Q(0_{\mathbb{R}^{\mathbb{Z}}}) = 0$, $\int_{\mathbb{R}^{\mathbb{Z}}} \left(x_0^2 \wedge 1\right) Q\left(\mathrm{d}\{x_n\}_{n\in\mathbb{Z}}\right) < \infty$ and

$$\begin{aligned}&\mathbb{E}\left[\exp\left(i\sum_{k=n_1}^{n_2} a_k X_k\right)\right]\\&= \exp\left[\int_{\mathbb{R}^{\mathbb{Z}}}\left(\exp\left(i\sum_{k=n_1}^{n_2} a_k x_k\right) - 1 - i\sum_{k=n_1}^{n_2} a_k c(x_k)\right) Q\left(\mathrm{d}\{x_n\}_{n\in\mathbb{Z}}\right)\right]\end{aligned}$$

for any choice of $-\infty < n_1 \le n_2 < +\infty$, $\{a_k\}_{n_1 \le n_2} \in \mathbb{R}^{n_2 - n_1}$.
c is defined by:
$c(x) = -1$ if $x < -1$
$c(x) = x$ if $-1 \le x \le 1$
$c(x) = 1$ if $x > 1$

The system $\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}, Q, S\right)$ where S is the shift on $\mathbb{R}^{\mathbb{Z}}$ is called the *Lévy measure system* associated to the process X.

The α-stable stationary processes, $0<\alpha<2$, are (see Chapter 3 in [17]) completely characterized as those IDp processes such that their Lévy measure satisfies

$$(R_t)_* Q = t^{-\alpha} Q \tag{5.1}$$

for any positive t, R_t being the multiplication by t, i.e.

$$\{x_n\}_{n\in\mathbb{Z}} \mapsto \{tx_n\}_{n\in\mathbb{Z}}.$$

We also recall the fundamental result of Maruyama that allows to represent any IDp process with Lévy measure Q as a stochastic integral with respect to a Poisson measure with intensity Q.

Theorem 6. *(Maruyama representation [8]) Let $\mathbb{P}$ be the distribution of a stationary IDp process with Lévy measure Q and $\left(\left(\mathbb{R}^{\mathbb{Z}}\right)^*, \left(\mathcal{B}^{\otimes\mathbb{Z}}\right)^*, Q^*, S_*\right)$ the Poisson measure over the Lévy measure system $\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}, Q, S\right)$. Set X_0 as $\{x_n\}_{n\in\mathbb{Z}} \mapsto x_0$ and define, on $\left(\mathbb{R}^{\mathbb{Z}}\right)^*$, the stochastic integral $I(X_0)$ as the limit in probability, as n tends to infinity, of the random variables*

$$\nu \mapsto \int_{|X_0|>\frac{1}{n}} X_0 d\nu - \int_{|X_0|>\frac{1}{n}} c(X_0)\, dQ.$$

Then the process $\{I(X_0)\circ S_*^n\}_{n\in\mathbb{Z}}$ has distribution $\mathbb{P}$.

5.2. **Lévy measure as Maharam system.**

Theorem 7. *Let $\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}, Q, S\right)$ be the Lévy measure system of an α-stable stationary process. Then there exists a probability space $(\Omega, \mathcal{F}, \mu)$, a non singular transformation T, a function $f\in L^\alpha(\mu)$ such that, if M denotes the map $(\omega,t)\mapsto tf(\omega)$ then the map $\Theta := (\omega,t)\mapsto \left\{M\circ \widetilde{T}_\alpha^n(\omega,t)\right\}_{n\in\mathbb{Z}}$ yields an isomorphism of the Maharam system $\left(\Omega\times\mathbb{R}_+, \mathcal{F}\otimes\mathcal{B}_+, \mu\otimes\frac{1}{s^{1+\alpha}}ds, \widetilde{T}_\alpha\right)$ with $\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}, Q, S\right)$.*

Proof. First observe that Theorem 3 can be applied to $\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}, Q, S\right)$ since the measurable and (obviously) dissipative flow $\left\{R_{\mathrm{e}^{\frac{t}{\alpha}}}\right\}_{t\in\mathbb{R}}$ satisfies the hypothesis, thanks to Eq (5.1). Therefore, there exists an isomorphism Ψ between the Maharam system $\left(\Omega\times\mathbb{R}_+, \mathcal{F}\otimes\mathcal{B}_+, \mu\otimes\frac{1}{s^{1+\alpha}}\mathrm{d}s, \widetilde{T}_\alpha\right)$ and $\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}, Q, S\right)$ for an appropriate non-singular system $(\Omega, \mathcal{F}, \mu, T)$. Set $f := \Psi(\omega,1)_0$ (i.e. $\Psi(\omega,1)_0$ is the 0-th coordinate of the sequence $\Psi(\omega,1)$) and let us check that $f\in L^\alpha(\mu)$. Indeed, as Q is a Lévy measure, we have:

$$\int_{\mathbb{R}^{\mathbb{Z}}} x_0^2 \wedge 1 Q\left(\mathrm{d}\{x_n\}_{n\in\mathbb{Z}}\right) < \infty$$

but since Ψ is an isomorphism and $\Psi(\omega,t) = \Psi\circ S_t(\omega,1) = R_t\circ\Psi(\omega,1) = t\Psi(\omega,1)$, we have:

$$\begin{aligned}\int_{\mathbb{R}^{\mathbb{Z}}} x_0^2 \wedge 1 Q\left(\mathrm{d}\left\{x_n\right\}_{n\in\mathbb{Z}}\right) &= \int_\Omega \int_{\mathbb{R}_+} \Psi\left(\omega,t\right)_0^2 \wedge 1 \frac{1}{t^{\alpha+1}} \mathrm{d}t \mu\left(\mathrm{d}\omega\right) \\ \int_\Omega \int_{\mathbb{R}_+} \left(t^2 \Psi\left(\omega,1\right)_0^2\right) \wedge 1 \frac{1}{t^{\alpha+1}} \mathrm{d}t \mu\left(\mathrm{d}\omega\right) &= \left(\int_{\mathbb{R}_+} z^2 \wedge 1 \frac{1}{z^{\alpha+1}} \mathrm{d}z\right) \int_\Omega \left|\Psi\left(\omega,1\right)_0\right|^\alpha \mu\left(\mathrm{d}\omega\right)\end{aligned}$$

after the change of variable $z := t\left|\Psi\left(\omega,1\right)_0\right|$. Therefore $\int_\Omega \left|\Psi\left(\omega,1\right)_0\right|^\alpha \mu\left(\mathrm{d}\omega\right) < \infty$. $\square$

In the symmetric case we can precise the theorem:

Theorem 8. *Let $\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}, Q, S\right)$ be the Lévy measure system of a symmetric α-stable stationary process. Then there exists a probability space $(X, \mathcal{A}, \nu)$, a non singular transformation R, a function $f \in L^\alpha(\nu)$ and a measurable map $\xi : X \to \{-1,1\}$ such that, if M denotes the map $(x,t) \mapsto tf(x)$ then the map $(x,t) \mapsto \left\{M \circ \overline{R_\alpha}^n (x,t)\right\}_{n\in\mathbb{Z}}$ yields an isomorphism between $\left(X \times \mathbb{R}^*, \mathcal{A} \otimes \mathcal{B}, \nu \otimes \frac{1}{|s|^{1+\alpha}} ds, \overline{T_\alpha}\right)$ with $\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}, Q, S\right)$, $\overline{R_\alpha}$ being defined by $(x,t) \mapsto \left(Rx, \xi(x) t \left(\frac{dR_*^{-1}\mu}{d\mu}(x)\right)^{\frac{1}{\alpha}}\right)$.*

Proof. Start by applying Theorem 7 to the Lévy measure.

Observe that the symmetry involves the presence of a measure preserving involution I, namely $I\{x_n\}_{n\in\mathbb{Z}} = \{-x_n\}_{n\in\mathbb{Z}}$. I also preserves the Lévy measure of the process. Observe also that I commutes with the shift and with the flow R_t. Therefore $\widetilde{I} := \Theta^{-1} I \Theta$ is a measure preserving automorphism of $\left(\Omega \times \mathbb{R}_+^*, \mathcal{F} \otimes \mathcal{B}, \mu \otimes \frac{1}{s^{1+\alpha}} \mathrm{d}s, \widetilde{T}\right)$ and we can apply Lemma 4 to deduce that $\widetilde{I}$ induces a non singular involution ϕ on $(\Omega, \mathcal{F}, \mu, T)$. It is standard that such transformation admits an equivalent finite invariant measure so, up to another measure preserving isomorphism, we can assume that ϕ preserves the probability measure μ.

Using Rohklin structure theorem, the compact factor associated to the compact group $\{\mathrm{Id}, \phi\}$ tells us that we can represent $(\Omega, \mathcal{F}, \mu, T)$ as $\left(X \times \{-1,1\}, \mathcal{A} \otimes \mathcal{P}\{-1,1\}, \nu \otimes m, R_\xi\right)$ where R is a non-singular automorphism of $(X, \mathcal{A}, \nu)$, m is the uniform measure on $(\{-1,1\}, \mathcal{P}\{-1,1\})$, ξ a cocyle from X to $\{-1,1\}$ and $R_\xi := (x,\epsilon) \mapsto (Rx, \xi(x)\epsilon)$.

It is now clear that $\left(X \times \{-1,1\} \times \mathbb{R}_+^*, (\mathcal{A} \otimes \mathcal{P}\{-1,1\}) \otimes \mathcal{B}, \nu \otimes m \otimes \frac{1}{s^{1+\alpha}} \mathrm{d}s, \widetilde{S_\xi}\right)$ is isomorphic to $\left(X \times \mathbb{R}^*, \mathcal{A} \otimes \mathcal{B}, \nu \otimes \frac{1}{|s|^{1+\alpha}} \mathrm{d}s, \overline{R_\alpha}\right)$ thanks to the mapping $(x,\epsilon,t) \mapsto \left(x, 2^{\frac{1}{\alpha}} \epsilon t\right)$ and $\overline{R_\alpha} := (x,t) \mapsto \left(Rx, \xi(x) \left(\frac{\mathrm{d}R_*^{-1}\mu}{\mathrm{d}\mu}(x)\right)^{\frac{1}{\alpha}} t\right)$. $\square$

5.3. **Spectral representation.** It is now very easy to derive spectral representations from the above results. In particular, if $\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}, Q, S\right)$ is the Lévy measure system of an $S\alpha S$ stationary process, under the notation of Theorem 8, $(X, \mathcal{A}, \nu)$ together with the function $f \in L^\alpha(\nu)$, the cocycle ϕ and the non-singular automorphism T yields a spectral representation of the process. Indeed, by building the Poisson measure over $\left(X \times \mathbb{R}^*, \mathcal{A} \otimes \mathcal{B}, \nu \otimes \frac{1}{|s|^{1+\alpha}} \mathrm{d}s, \overline{R_\alpha}\right)$ and by applying to it, f and ξ Theorem 3.12.2, page 156 in [16], we recover the $S\alpha S$ process with Lévy measure Q, which proves the validity of the spectral representation. The minimality can be obtained without difficulty thanks to Proposition 2.2 in [10].

5.4. **Maharam systems as Lévy measure.** We can ask whether if a Maharam system $\left(\Omega \times \mathbb{R}_+, \mathcal{F} \otimes \mathcal{B}_+, \mu \otimes \frac{1}{s^{1+\alpha}} \mathrm{d}s, \widetilde{T}_\alpha\right)$ can be coded into a Lévy measure system of a stable process. We can answer affirmatively this question in the only interesting case, that is when the Maharam system has no finite absolutely continuous $\widetilde{T}_\alpha$-invariant measure, that is, when the resulting Lévy measure system leads to an ergodic stable process.

Recall that a Maharam system has no finite absolutely continuous $\widetilde{T}_\alpha$-invariant measure if and only if the non-singular system $(\Omega, \mathcal{F}, \mu, T)$ has the same property. But then a famous theorem of Krengel [7] shows that such a system possesses a 2-generator, that is, there exists a measurable function $f : \Omega \to \{0,1\}$, $\mu\{f = 1\} < \infty$, such that $\sigma\{f \circ T^n, n \in \mathbb{Z}\} = \mathcal{F}$.

To be more precise, this means that, up to isomorphism, these Maharam systems can be represented as $\left(\{0,1\}^{\mathbb{Z}} \times \mathbb{R}_+^*, \mathcal{B}\left(\{0,1\}^{\mathbb{Z}}\right) \otimes \mathcal{B}, \mu \otimes \frac{1}{s^{1+\alpha}} \mathrm{d}s, \overline{T_\alpha}\right)$ for an appropriate measure μ. But if φ is the map $\left(\{x_n\}_{n\in\mathbb{Z}}, t\right) \mapsto \{t x_n\}_{n\in\mathbb{Z}}$ and $Q = \varphi_*\left(\mu \otimes \frac{1}{s^{1+\alpha}}\right)$, we obtain a Lévy measure system of an α-stable system $\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}, Q, S\right)$ as $\{x_n\}_{n\in\mathbb{Z}} \mapsto \{x_0\}$ is in $L^\alpha(\mu)$ (see the proof of Theorem 8). Moreover, the sequence $\{y_n\}_{n\in\mathbb{Z}}$ takes only two values, 0 or $\sup\{y_n\}_{n\in\mathbb{Z}}$ Q-almost everywhere ($\{y_n\}_{n\in\mathbb{Z}}$ can't be identically zero as it would violate the II_∞ property), therefore, φ^{-1} exists and is defined by $\{y_n\}_{n\in\mathbb{Z}} \mapsto \left(\sup\{y_n\}_{n\in\mathbb{Z}}, \left\{\frac{y_n}{\sup\{y_n\}_{n\in\mathbb{Z}}}\right\}_{n\in\mathbb{Z}}\right)$.

6. Refinements of the representation

Ergodic stationary processes are building blocks of stationary processes, prime numbers are the building blocks of integers, factors are building blocks of Von Neumannn algebras etc. What are the building blocks of stationary infinitely divisible processes ? Let's get more precise:

Given a stationary ID process X, what are the solutions to the equation (in distribution):

$$X = X_1 + X_2$$

where X_1 and X_2 are independent stationary ID processes. Of course, if Q is the Lévy measure of X, then taking X_1 with Lévy measure $c_1 Q$ and X_2 with Lévy measure $c_2 Q$ with $c_1 + c_2 = 1$ gives a solution. If these are the only solutions, we said in [12] that X is *pure*, meaning that is impossible to reduce X to "simpler" pieces. It was then very easy to show that X is pure if and only if its Lévy measure is ergodic:

Proposition 9. *A stationary IDp process X is pure if and only if its Lévy measure Q is ergodic*

Proof. Assume Q is not ergodic. There exists a partition of $\mathbb{R}^{\mathbb{Z}}$ into two shift invariant sets A and B both of positive measure. Therefore, $Q_{|A}$ and $Q_{|B}$ can be taken as Lévy measures of two stationary IDp processes X_A and X_B and taking them independent leads to

$$X = X_A + X_B$$

in distribution, as $Q = Q_{|A} + Q_{|B}$.

In the converse, assume Q is ergodic and suppose there exist independent stationary IDp processes X_1 and X_2 with Lévy measure Q_1 and Q_2 such that

$$X = X_1 + X_2$$

holds in distribution. As $Q = Q_1 + Q_2$ we get $Q_1 \ll Q$. But as Q is ergodic, this in turns implies that there exists $c > 0$ such that $Q_1 = cQ$ and thus $Q_2 = (1-c)\,Q$. □

In this section, we will try to comment the above equation according to the Krieger type of the associated non-singular transformation. A description of the interesting class of those stable processes driven by non-singular transformations of type III_0 is unknown.

6.1. **The type III_1 case, pure stable processes.** It was an open question whether there exist pure stable processes. It can now be solved thanks to the Maharam structure of the Lévy measure: an α-stable process is pure if and only if the underlying non-singular system is of type III_1.

The existence of pure stable processes (guaranteed by the comments made in Section 5.4) is reassuring as it validates the specific study of stable processes.

6.2. **The type III_λ case, $0 < \lambda < 1$.** In this section we derive the form of those α-stable processes associated with an ergodic, type III_λ non-singular automorphism, $0 < \lambda < 1$.

6.2.1. *Semi-stable stationary processes.* An infinitely divisible probability measure μ on $\mathbb{R}^d$ is called *α-semi-stable* with *span b* if its Fourier transform satisfies:

$$\hat{\mu}(z)^{b^\alpha} = \hat{\mu}(bz)\, e^{i\langle c, z\rangle}$$

for some $c \in \mathbb{R}^d$.

By extension, an α-semi-stable process process is a process whose finite dimensional distributions are α-semi-stable. Using once again results of Chapter 3 in [17], one gets the following characterization of α-semi-stable stationary processes:

A shift-invariant Lévy measure Q on $\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}\right)$ is the Lévy measure of an α-semi-stable stationary process of span $b > 0$ if and only if it satisfies

$$(R_b)_* Q = b^{-\alpha} Q$$

where R_b is the multiplication by b

$$\{x_n\}_{\in\mathbb{Z}} \mapsto \{bx_n\}_{\in\mathbb{Z}}.$$

Of course by iterating R_b, we easily observe that $(R_{b^n})_* Q = b^{-n\alpha} Q$ for all $n \in \mathbb{Z}$.

6.2.2. *Discrete Maharam extension.* Assume $(\Omega, \mathcal{F}, \mu, T)$ is a non-singular system such that there exists $\lambda > 0$ so that $\frac{\mathrm{d}T_*^{-1}\mu}{\mathrm{d}\mu} \in \{\lambda^n,\, n \in \mathbb{Z}\}$ μ-almost everywhere. We can form its *discrete Maharam extension*, that is, the m.p. system $\left(\Omega \times \mathbb{Z}, \mathcal{F} \otimes \mathcal{B}, \mu \otimes \lambda^n \mathrm{d}n, \widetilde{T}\right)$ where $\lambda^n \mathrm{d}n$ stands for the measure $\sum_{n\in\mathbb{Z}} \lambda^n \delta_n$ on $(\mathbb{Z}, \mathcal{B})$ and $\widetilde{T}$ is defined by

$$\widetilde{T}(\omega, n) = \left(T\omega, n - \log_\lambda \frac{\mathrm{d}T_*^{-1}\mu}{\mathrm{d}\mu}(\omega)\right).$$

6.2.3. *Ergodic decomposition of Maharam extension of type* III_λ *transformations.* Let $(\Omega, \mathcal{F}, \mu, T)$ be an ergodic type III_λ system. Up to a change of measure we can assume that the Radon Nykodim derivative take its values in the group $\{\lambda^n,\, n \in \mathbb{Z}\}$ where $r(T) = \{0, \lambda^n,\, n \in \mathbb{Z}, +\infty\}$ is the ratio set of T (see [6]). Therefore, the discrete Maharam extension $\left(\Omega \times \mathbb{Z}, \mathcal{F} \otimes \mathcal{B}, \beta\mu \otimes \lambda^n \mathrm{d}n, \widetilde{T}\right)$, where $\beta := \int_0^{-\ln\lambda} \mathrm{e}^{-s}\mathrm{d}s$, exists. Now form the product system

$$\left(\Omega \times \mathbb{Z} \times [0, -\ln\lambda[\,, \mathcal{F} \otimes \mathcal{B} \otimes \mathcal{B}\left([0, -\ln\lambda[\right), \beta\mu \otimes \lambda^n \mathrm{d}n \otimes \frac{\mathrm{e}^{-s}}{\beta}\mathrm{d}s, \widetilde{T} \times \mathrm{Id}\right).$$

The dissipative non singular flow $S_t : (\omega, n, s) \mapsto \left(\omega, n + \left\lfloor \frac{s-t}{-\ln\lambda} \right\rfloor, s - t + \ln\lambda \left\lfloor \frac{s-t}{-\ln\lambda} \right\rfloor\right)$ satisfies $S_t \circ \widetilde{T} \times \mathrm{Id} = \widetilde{T} \times \mathrm{Id} \circ S_t$ and $(S_t)_* \mu \otimes \lambda^n \mathrm{d}n \otimes \mathrm{e}^{-s}\mathrm{d}s = \mathrm{e}^{-t}\mu \otimes \lambda^n \mathrm{d}n \otimes \mathrm{e}^{-s}\mathrm{d}s$ and it is very easy to see that $(\mathbb{Z} \times [0, -\ln\lambda[\,, \mathcal{B} \otimes \mathcal{B}\left([0, -\ln\lambda[\right), \lambda^n \mathrm{d}n \otimes \mathrm{e}^{-s}\mathrm{d}s)$ is just a reparametrization of $(\mathbb{R}, \mathcal{B}, \mathrm{e}^s \mathrm{d}s)$ thanks to the mapping $(n, s) \mapsto -n\ln\lambda - s$.

Therefore $\left(\Omega \times \mathbb{Z} \times [0, -\ln\lambda[\,, \mathcal{F} \otimes \mathcal{B} \otimes \mathcal{B}\left([0, -\ln\lambda[\right), \mu \otimes \lambda^n \mathrm{d}n \otimes \mathrm{e}^{-s}\mathrm{d}s, \widetilde{T} \times \mathrm{Id}\right)$ can be seen as the Maharam extension of $(\Omega, \mathcal{F}, \mu, T)$.

It remains to prove the ergodicity of $\left(\Omega \times \mathbb{Z}, \mathcal{F} \otimes \mathcal{B}, \beta\mu \otimes \lambda^n \mathrm{d}n, \widetilde{T}\right)$, this follows, for example, from Corollary 5.4 in [18], as the ratio set is precisely the set of essential values of Radon-Nykodim cocycle.

We then obtain the ergodic decomposition of the Maharam extension: it is the discrete Maharam extension $\left(\Omega \times \mathbb{Z}, \mathcal{F} \otimes \mathcal{B}, \beta\mu \otimes \lambda^n \mathrm{d}n, \widetilde{T}\right)$ randomized by the measure $\frac{\mathrm{e}^{-s}}{\beta}\mathrm{d}s$ on $[0, -\ln\lambda[$.

6.2.4. *Application to stable processes.* Let $\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}, Q, S\right)$ be the Lévy measure system of an α-stable process driven by an ergodic type III_λ system $(\Omega, \mathcal{F}, \mu, T)$ and let $f \in L^\alpha(\mu)$ be given as in Theorem 7. Let $b > 1$ so that $b^{-\alpha} = \lambda$, we need to obtain a multiplicative version of the above structure adapted to our parameters. Up to a change of measure we can assume that $\left(\frac{\mathrm{d}T_*^{-1}\mu}{\mathrm{d}\mu}\right)^{\frac{1}{\alpha}} \in \{b^n,\, n \in \mathbb{Z}\}$ μ-almost everywhere. Consider the discrete Maharam extension $\left(\Omega \times G_b, \mathcal{F} \otimes \mathcal{B}, \beta\mu \otimes m_b, \widetilde{T}\right)$ (in a multiplicative representation) where $\beta = \int_1^b \frac{1}{s^{1+\alpha}}\mathrm{d}s$, m_b is the measure $\sum_{g \in G_b} g^{-\alpha}\delta_g$ on $G_b := \{b^n,\, n \in \mathbb{Z}\}$ and $\widetilde{T} := (\omega, g) \mapsto \left(T\omega, g\left(\frac{\mathrm{d}T_*^{-1}\mu}{\mathrm{d}\mu}(\omega)\right)^{\frac{1}{\alpha}}\right)$. Form the system $\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}, Q^r, S\right)$ as a factor of $\left(\Omega \times G_b, \mathcal{F} \otimes \mathcal{B}, \beta\mu \otimes m_b, \widetilde{T}\right)$ given by the mapping $\varphi := (\omega, g) \mapsto \left\{M \circ \widetilde{T}^n(\omega, g)\right\}_{n \in \mathbb{Z}}$ where $M(\omega, g) = gf(\omega)$ and $Q^r = \varphi_*(\beta\mu \otimes m_b)$.

Now, as above, we recover the Maharam extension of $(\Omega, \mathcal{F}, \mu, T)$ by considering $\left(\Omega \times G_b \times [1, b[\,, \mathcal{F} \otimes \mathcal{B} \otimes \mathcal{B}([1, b[), \beta\mu \otimes m_b \otimes \frac{1}{\beta s^{1+\alpha}}\mathrm{d}s, \widetilde{T} \times \mathrm{Id}\right)$. As the system $\left(G_b \times [1, b[\,, \mathcal{B} \otimes \mathcal{B}([1, b[), m_b \otimes \frac{1}{s^{1+\alpha}}\mathrm{d}s\right)$ is isomorphic to $\left(\mathbb{R}_+^*, \mathcal{B}, \frac{1}{s^{1+\alpha}}\mathrm{d}s\right)$ thanks to $(g, t) \mapsto gt$, we obtain $\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}, Q, S\right)$ by applying the map $\left(\{x_n\}_{n \in \mathbb{Z}}, t\right) \mapsto \{tx_n\}_{n \in \mathbb{Z}}$ to $\left(\mathbb{R}^{\mathbb{Z}} \times [1, b[\,, \mathcal{B}^{\otimes\mathbb{Z}} \otimes \mathcal{B}([1, b[), Q^r \otimes \frac{1}{\beta s^{1+\alpha}}\mathrm{d}s, S \times \mathrm{Id}\right)$. At last, we can check that Q^r is a Lévy measure, indeed we know that

$$\int_{\mathbb{R}^{\mathbb{Z}}} \left(x_0^2 \wedge 1\right) Q\left(\mathrm{d}\left\{x_n\right\}_{n\in\mathbb{Z}}\right) < +\infty$$

but

$$\int_{\mathbb{R}^{\mathbb{Z}}} \left(x_0^2 \wedge 1\right) Q\left(\mathrm{d}\left\{x_n\right\}_{n\in\mathbb{Z}}\right) = \int_1^b \left(\int_{\mathbb{R}^{\mathbb{Z}}} \left(\left(sx_0\right)^2 \wedge 1\right) Q^r\left(\mathrm{d}\left\{x_n\right\}_{n\in\mathbb{Z}}\right)\right) \frac{1}{\beta s^{1+\alpha}}\mathrm{d}s$$

therefore, for some $1 \le s < b$, $\int_{\mathbb{R}^{\mathbb{Z}}} \left(\left(sx_0\right)^2 \wedge 1\right) Q^r\left(\mathrm{d}\left\{x_n\right\}_{n\in\mathbb{Z}}\right) < +\infty$ and this is enough to prove that Q^r is a Lévy measure.

$\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}, Q^r, S\right)$ is the Lévy measure system of an α-semi-stable stationary process with span b. Heuristically, if X has Lévy measure Q, X can be thought as the continuous sum of independent processes Y^t, $1 \le t < b$ weighted by the probability measure $\frac{1}{\beta s^{1+\alpha}}\mathrm{d}s$ where $\frac{1}{t}Y^t$ has Lévy measure Q^r. More formally, if $\left(\left(\Omega \times G_b \times [1,b[\right)^*, \left(\mathcal{F}\otimes\mathcal{B}\otimes\mathcal{B}\left([1,b[\right)\right)^*, \left(\beta\mu\otimes m_b\otimes\frac{1}{\beta s^{1+\alpha}}\mathrm{d}s\right)^*, \left(\widetilde{T}\times\mathrm{Id}\right)_*\right)$ denotes the Poisson suspension over $\left(\Omega \times G_b \times [1,b[, \mathcal{F}\otimes\mathcal{B}\otimes\mathcal{B}\left([1,b[\right), \beta\mu\otimes m_b\otimes\frac{1}{\beta s^{1+\alpha}}\mathrm{d}s, \widetilde{T}\times\mathrm{Id}\right)$, then, if I denotes the stochastic integral as in Theorem 6, $X := \left\{I\left\{M_1\right\}\circ\left(\widetilde{T}\times\mathrm{Id}\right)_*^n\right\}_{n\in\mathbb{Z}}$ has Lévy measure Q and $Y := \left\{I\left\{M_2\right\}\circ\left(\widetilde{T}\times\mathrm{Id}\right)_*^n\right\}_{n\in\mathbb{Z}}$ where $M_1\left(\omega,g,s\right) = sgf\left(\omega\right)$ and $M_2\left(\omega,g,s\right) = gf\left(\omega\right)$.

We therefore observe that X is entirely determined by a pure α-semi-stable stationary process with span b, Y. It is very easy to see that X and Y share the same type of mixing.

6.2.5. *Examples.* It is not difficult to exhibit examples of stable processes of the kind described above as the structure detailed allows to build such processes. We can, for example, consider the systems T_p, $\frac{1}{2} < p < 1$ introduced in [4]. We will follow the presentation given in [1] (page 104).

Let Ω be the group of dyadic integers, let τ acts by translation by $\underline{1}$ on Ω and for $\frac{1}{2} < p < 1$, let μ_p be a probability measure on Ω defined on cylinders by

$$\mu_p\left(\left[\epsilon_1,\ldots,\epsilon_n\right]\right) = \prod_{k=1}^{n} p\left(\epsilon_k\right),$$

where $p\left(0\right) = 1-p$ and $p\left(1\right) = p$.

If we set $\frac{1-p}{p} = \lambda$, we get:

$$\frac{\mathrm{d}\tau_*^{-1}\mu_p}{\mathrm{d}\mu_p} = \lambda^{\phi}$$

where $\phi\left(x\right) = \min\left\{n\in\mathbb{N},\, x_n = 0\right\} - 2$. It is proved in [4] that the discrete Maharam extension $\left(\Omega\otimes\mathbb{Z}, \mathcal{F}\otimes\mathcal{B}, \mu\otimes\lambda^n\mathrm{d}n, \widetilde{\tau}\right)$ is ergodic.

We can form a new system, which will be the Lévy measure system of a stationary semi-stable process with span λ^{α}, thanks to the following map

$$f : \left(\omega, n\right) \mapsto \lambda^{\alpha n} \sum_{i\ge 1} \omega_i 2^{-i}$$

The Lévy measure Q^r is the image of $\mu\otimes\lambda^n\mathrm{d}n$ by the map $\left(\omega,n\right) \mapsto \left\{f\circ\widetilde{\tau}^k\left(\omega,n\right)\right\}_{k\in\mathbb{Z}}$.

By randomizing this Lévy measure as explained above, we obtain the Lévy measure Q of a stationary α-stable process, that is Q is the image measure of $Q^r \otimes \frac{1}{\beta s^{1+\alpha}} \mathrm{d}s$ by the map

$$\left(\{x_n\}_{n\in\mathbb{Z}}, t\right) \mapsto \{tx_n\}_{n\in\mathbb{Z}}.$$

To obtain a realization of these two processes as stochastic integrals over Poisson suspensions, we can proceed as explained at the end of the preceding section.

Anticipating next sections, we derive the ergodic properties of these processes:

τ being of type III_λ, the Maharam extension is of type II_∞ which means that the Lévy measure system of the corresponding stationary α-stable process (with Lévy measure Q) is of type II_∞. Therefore the associated Poisson suspension is weakly mixing. As stochastic integrals with respect to this Poisson suspension, both processes (with Lévy measures Q and Q^r) are weakly mixing.

Thanks to Lemma 1.2.10 page 30 in [1], τ is rigid for the sequence $\{2^n\}_{n\in\mathbb{N}}$. Therefore, by Theorem 18 (or with a slight adaptation for the semi-stable case), both processes are also rigid for the same sequence.

6.3. **The type** I **and** II **cases.** This case is easy to deal with as we can assume the associated ergodic nonsingular system is actually measure preserving, that is, the Lévy measure system $\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}, Q, S\right)$ is isomorphic to $\left(\Omega \times \mathbb{R}_+^*, \mathcal{F}\otimes\mathcal{B}, \mu\otimes\frac{1}{s^{1+\alpha}}\mathrm{d}s, \widetilde{T}\right)$ where T preserves μ and $\widetilde{T}$ acts as $T\times \mathrm{Id}$, i.e. $\widetilde{T}(\omega,t) = (T\omega, t)$. Considering $f \in L^\alpha(\mu)$ furnished by Theorem 7, $\left(\Omega \times \mathbb{R}_+^*, \mathcal{F}\otimes\mathcal{B}, \mu\otimes\frac{1}{s^{1+\alpha}}\mathrm{d}s, \widetilde{T}\right)$ is isomorphic to $\left(\mathbb{R}^{\mathbb{Z}}\times\mathbb{R}_+^*, \mathcal{B}^{\otimes\mathbb{Z}}\otimes\mathcal{B}, Q^s\otimes\frac{1}{s^{1+\alpha}}\mathrm{d}s, S\times\mathrm{Id}\right)$ through the map $(\omega,t)\mapsto \left(\{f\circ T^n(\omega)\}_{n\in\mathbb{Z}}, t\right)$ and

$$\begin{aligned}
\int_{\mathbb{R}^{\mathbb{Z}}} \left(x_0^2\wedge 1\right) Q\left(\mathrm{d}\{x_n\}_{n\in\mathbb{Z}}\right) &= \int_\Omega\int_{\mathbb{R}_+} \left((tf(\omega))^2\wedge 1\right)\frac{1}{t^{\alpha+1}}\mathrm{d}t\mu(\mathrm{d}\omega)\\
&= \int_{\mathbb{R}_+}\left(\int_{\mathbb{R}^{\mathbb{Z}}}\left((tx_0)^2\wedge 1\right) Q^s\left(\mathrm{d}\{x_n\}_{n\in\mathbb{Z}}\right)\right)\frac{1}{t^{\alpha+1}}\mathrm{d}t < +\infty
\end{aligned}$$

For the same reason as above Q^s is a Lévy measure. We draw the same conclusions as in the preceding section taking into account that the weight is now the infinite measure $\frac{1}{t^{\alpha+1}}\mathrm{d}t$ on $\mathbb{R}_+^*$ and Q^s can be any Lévy measure (of a stationary IDp process).

7. Ergodic properties

Some ergodic properties of general IDp stationary processes have been given in terms of ergodic properties of the Lévy measure system in [13]. For an α-stable stationary processes, it is more interesting to give them in terms of the associated non-singular system $(\Omega, \mathcal{F}, \mu, T)$. This work has been undertaken in the symmetric ($S\alpha S$) case in a series of papers (see in particular [11] and [15]).

We have a new tool to deal with this problem:

As the Lévy measure of an α-stable stationary processes can now be seen as the Maharam extension $\left(\Omega \times \mathbb{R}_+^*, \mathcal{F}\otimes\mathcal{B}, \mu\otimes\frac{1}{s^{1+\alpha}}\mathrm{d}s, \widetilde{T}\right)$ of the system $(\Omega, \mathcal{F}, \mu, T)$, it suffices to connect ergodic properties of T and $\widetilde{T}$, then apply the general results relating ergodic properties of a stationary IDp process with respect to those of its Lévy measure system.

Observe that linking ergodic properties of T and $\widetilde{T}$ is a general problem in non-singular ergodic theory which is of great interest.

We'll illustrate this in the following sections dealing with mixing, K-property and rigidity, the last two having been neglected in the α-stable literature.

7.1. **Mixing.** First recall that if S is a non singular transformation of a measure space $(X, \mathcal{A}, m)$, it induces a unitary operator U_S on $L^2(m)$ by

$$U_S f(x) = \sqrt{\frac{\mathrm{d}S_*^{-1}\mu}{\mathrm{d}\mu}(x)} f \circ S(x)$$

We first prove a general result, whose proof, that we sketch here, can be extracted from [11]:

Proposition 10. *The Maharam system* $\left(\Omega \times \mathbb{R}_+^*, \mathcal{F} \otimes \mathcal{B}_+, \mu \otimes \frac{1}{s^{1+\alpha}} ds, \widetilde{T}_\alpha\right)$ *is of zero type if and only if for all* $f \in L^2(\mu)$, $\langle U_T^n f, f\rangle_{L^2(\mu)} \to 0$ *as* n *tends to infinity.*

Proof. The "if" part follows from Cauchy-Schwarz inequality which leads to:

$$\left\langle U_{\widetilde{T}_\alpha}^n f \otimes g, f \otimes g \right\rangle_{L^2\left(\mu \otimes \frac{1}{s^{1+\alpha}}\mathrm{d}s\right)} \leq \|g\|_2^2 \langle U_T^n f, f\rangle_{L^2(\mu)},$$

for $f \in L^2(\mu)$ and $g \in L^2\left(\frac{1}{s^{1+\alpha}}\mathrm{d}s\right)$.

The "only if" part can be obtain by assuming that μ is a probability measure and by considering $g(s) = s^{\frac{\alpha}{2}-\epsilon} 1_{s\geq 1}$ and $g'(s) = s^{\frac{\alpha}{2}-\epsilon} 1_{s\geq c}$. A direct computation, using Jensen inequality with the concave function $x \mapsto x^{\frac{1}{1+\frac{2\epsilon}{\alpha}}}$ gives:

$$\left(2\epsilon \left\langle U_{\widetilde{T}_\alpha}^n 1 \otimes g, 1 \otimes g' \right\rangle_{L^2\left(\mu \otimes \frac{1}{s^{1+\alpha}}\mathrm{d}s\right)}\right)^{\frac{1}{1+\frac{2\epsilon}{\alpha}}} \geq \int_\Omega \sqrt{\frac{\mathrm{d}T_*^{-n}\mu}{\mathrm{d}\mu}(\omega)} 1_{\sqrt{\frac{\mathrm{d}T_*^{-n}\mu}{\mathrm{d}\mu}(\omega)} \leq c^{-\alpha}} \mu(\mathrm{d}\omega)$$

Moreover, as $\int_\Omega \sqrt{\frac{\mathrm{d}T_*^{-n}\mu}{\mathrm{d}\mu}(\omega)} 1_{\sqrt{\frac{\mathrm{d}T_*^{-n}\mu}{\mathrm{d}\mu}(\omega)} > c^{-\alpha}} \mu(\mathrm{d}\omega) \leq c^{\frac{\alpha}{2}}$ (thanks to Cauchy-Schwarz and Markov inequality), we get:

$$\int_\Omega \sqrt{\frac{\mathrm{d}T_*^{-n}\mu}{\mathrm{d}\mu}(\omega)} \mu(\mathrm{d}\omega) \leq \int_\Omega \sqrt{\frac{\mathrm{d}T_*^{-n}\mu}{\mathrm{d}\mu}(\omega)} 1_{\sqrt{\frac{\mathrm{d}T_*^{-n}\mu}{\mathrm{d}\mu}(\omega)} \leq c^{-\alpha}} \mu(\mathrm{d}\omega) + c^{\frac{\alpha}{2}}$$

Therefore, if $\widetilde{T}_\alpha$ is of zero type, $\int_\Omega \sqrt{\frac{\mathrm{d}T_*^{-n}\mu}{\mathrm{d}\mu}(\omega)} \mu(\mathrm{d}\omega)$ goes to zero as n tends to infinity which is sufficient to prove that $\langle U_T^n f, f\rangle_{L^2(\mu)} \to 0$ as n tends to infinity, for all $f \in L^2(\mu)$. $\square$

Combining this result with the characterization of the Lévy measure system as a Maharam system and the mixing criteria found in [13], we obtain the following theorem, already known in the $S\alpha S$-case (see [11]):

Theorem 11. *A stationary stable process* $\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}, \mathbb{P}, S\right)$ *with associated system* $(\Omega, \mathcal{F}, \mu, T)$ *is mixing if and only if for all* $f \in L^2(\mu)$, $\langle U_T^n f, f\rangle_{L^2(\mu)} \to 0$ *as* n *tends to infinity.*

In the forthcoming sections, we are interested in less known ergodic properties (K property and rigidity) that have been neglected in the α-stable literature.

7.2. K **property.**

Definition 12. (see [19]) A conservative non-singular system $(\Omega, \mathcal{F}, \mu, T)$ is a K-system if there exists a sub-σ-algebra $\mathcal{G} \subset \mathcal{F}$ such that $T^{-1}\mathcal{G} \subset \mathcal{G}$, $T^{-n}\mathcal{G} \downarrow \{\Omega, \emptyset\}$, $T^n\mathcal{G} \uparrow \mathcal{F}$ and $\frac{\mathrm{d}\mu}{\mathrm{d}T_*\mu}$ is $\mathcal{G}$-measurable.

A K-system is always ergodic (see [19]) .

Definition 13. A measure-preserving system $(X, \mathcal{A}, m, S)$ is remotely infinite if there exists a sub-σ-algebra $\mathcal{C} \subset \mathcal{A}$ such that $T^{-1}\mathcal{C} \subset \mathcal{C}$, $S^n\mathcal{C} \uparrow \mathcal{A}$ and $\bigcap_{n\geq 1} S^{-n}\mathcal{C}$ contains zero or infinite measure sets only.

Proposition 14. *If $(\Omega, \mathcal{F}, \mu, T)$ is a K-system which is not of type II_1 then its Maharam extension $\left(\Omega \times \mathbb{R}_+^*, \mathcal{F} \otimes \mathcal{B}_+, \mu \otimes \frac{1}{s^{1+\alpha}}ds, \widetilde{T}_\alpha\right)$ is remotely infinite.*

Proof. Let $\mathcal{G}$ be as in Definition 12. Observe that, as $\frac{\mathrm{d}\mu}{\mathrm{d}T_*\mu}$ is $\mathcal{G}$-measurable, $\mathcal{G} \otimes \mathcal{B}_+$ is $\widetilde{T}_\alpha$-invariant, that is $\widetilde{T}_\alpha^{-1}\mathcal{G} \otimes \mathcal{B}_+ \subset \mathcal{G} \otimes \mathcal{B}_+$. Indeed, take g $\mathcal{G}$-measurable and f $\mathcal{B}_+$-measurable, we get:

$$g \otimes f\left(\widetilde{T}_\alpha(\omega, s)\right) = \left(g(T\omega), s\left(\frac{\mathrm{d}T_*^{-1}\mu}{\mathrm{d}\mu}(\omega)\right)^{\frac{1}{\alpha}}\right) = \left(g(T\omega), s\left(\frac{\mathrm{d}\mu}{\mathrm{d}T_*\mu}(T\omega)\right)^{\frac{1}{\alpha}}\right).$$

We are going to show that $\mathcal{P} := \bigcap_{n\in\mathbb{N}} \widetilde{T}_\alpha^{-n}\mathcal{G} \otimes \mathcal{B}_+$ only contains sets of zero or infinite measure. Observe that, as S_t commutes with $\widetilde{T}_\alpha$ and preserves $\mathcal{G} \otimes \mathcal{B}_+$ for all $t > 0$ then , $S_t^{-1}\left(\widetilde{T}_\alpha^{-n}\mathcal{G} \otimes \mathcal{B}_+\right) \subset \widetilde{T}_\alpha^{-n}\mathcal{G} \otimes \mathcal{B}_+$ and therefore $S_t^{-1}\mathcal{P} \subset \mathcal{P}$ for all $t > 0$. Now consider the measurable union, say K, of $\mathcal{P}$-measurable sets of finite and positive measure. It is a $\widetilde{T}_\alpha$-invariant set and a S_t-invariant set as well. Recall that the non-singular action of the flow S_t on the ergodic components of $\left(\Omega \times \mathbb{R}_+^*, \mathcal{F} \otimes \mathcal{B}_+, \mu \otimes \frac{1}{s^{1+\alpha}}\mathrm{d}s, \widetilde{T}_\alpha\right)$ is ergodic, therefore, if $K \neq \emptyset$ then $K = \Omega \times \mathbb{R}_+^*$ mod. $\nu \otimes \frac{1}{s^{1+\alpha}}\mathrm{d}s$.

Assume $K = \Omega \times \mathbb{R}_+^*$, this implies that the measure $\mu \otimes \frac{1}{s^{1+\alpha}}\mathrm{d}s$ is σ-finite on $\mathcal{P}$ and therefore $\mathcal{P}$ is a factor of $\left(\Omega \times \mathbb{R}_+^*, \mathcal{F} \otimes \mathcal{B}_+, \mu \otimes \frac{1}{s^{1+\alpha}}\mathrm{d}s, \widetilde{T}_\alpha\right)$. Now consider the quotient space $\left(\Omega \times \mathbb{R}_+^*\right)_{/\mathcal{P}}$ that we can endow, with a slight abuse of notation with the σ-algebra $\mathcal{P}$. Let ρ be the image measure of $\mu \otimes \frac{1}{s^{1+\alpha}}\mathrm{d}s$ by the projection map π. On $\left(\left(\Omega \times \mathbb{R}_+^*\right)_{/\mathcal{P}}, \mathcal{P}, \rho\right)$ $\widetilde{T}_\alpha$ and the dissipative flow S_t induce a transformation U and a dissipative flow Z_t that satisfy

$$\pi \circ \widetilde{T}_\alpha = U \circ \pi,\ \pi \circ S_t = U \circ \pi \text{ and } U \circ Z_t = Z_t \circ U$$

Of course, thanks to Theorem 3, $\left(\left(\Omega \times \mathbb{R}_+^*\right)_{/\mathcal{P}}, \mathcal{P}, \rho, U\right)$ is a Maharam system, therefore, we can represent it as $\left(Y \times \mathbb{R}_+^*, \mathcal{K} \otimes \mathcal{B}_+, \sigma \otimes \frac{1}{s^{1+\alpha}}\mathrm{d}s, \widetilde{L}_\alpha\right)$ for a non-singular system $(Y, \mathcal{K}, \sigma, L)$. Applying Lemma 4, π induces a non-singular factor map Γ from $(\Omega, \mathcal{G}, \mu, T)$ to $(Y, \mathcal{K}, \sigma, L)$, which means that there exists a R-invariant σ-algebra $\mathcal{Z} \subset \mathcal{G}$ such that $\Gamma^{-1}\mathcal{K} = \mathcal{Z}$. But we can observe, that for all $n > 0$, the factor $\widetilde{T}_\alpha^{-n}\mathcal{G} \otimes \mathcal{B}_+$ corresponds to a Maharam system that corresponds to the factor $T^{-n}\mathcal{G}$ of $(\Omega, \mathcal{G}, \mu, T)$. Therefore, for all $n > 0$, $\mathcal{Z} \subset T^{-n}\mathcal{G}$,

i.e. $\mathcal{Z} \subset \bigcap_{n\in\mathbb{N}} T^{-n}\mathcal{G} = \{\Omega, \emptyset\}$. This means that $\mathcal{K} = \{Y, \emptyset\}$, or, in other words, that $(Y, \mathcal{K}, \sigma, L)$ is the trivial (one point) system. $\left(Y \times \mathbb{R}_+^*, \mathcal{K} \otimes \mathcal{B}_+, \sigma \otimes \frac{1}{s^{1+\alpha}}\mathrm{d}s, \widetilde{L}_\alpha\right)$ then possesses lots of invariant sets of positive finite measure, for example $A := Y \times [1, 2]$. But $\pi^{-1}(A)$ is in turn a positive and finite measure invariant set for the system $\left(\Omega \times \mathbb{R}_+^*, \mathcal{F} \otimes \mathcal{B}_+, \mu \otimes \frac{1}{s^{1+\alpha}}\mathrm{d}s, \widetilde{T}_\alpha\right)$ and the existence of such set is impossible in a Maharam extension of an ergodic system which doesn't posses a finite T-invariant probability measure $\nu \ll \mu$. We can conclude that $K = \emptyset$.

To prove that $\left(\Omega \times \mathbb{R}_+^*, \mathcal{F} \otimes \mathcal{B}_+, \mu \otimes \frac{1}{s^{1+\alpha}}\mathrm{d}s, \widetilde{T}_\alpha\right)$ is remotely infinite, it remains to show that $\bigvee_{n\in\mathbb{Z}} \widetilde{T}_\alpha^{-n}\mathcal{G} \otimes \mathcal{B}_+ = \mathcal{F} \otimes \mathcal{B}_+$. We only sketch the proof which consists into verifying that the operation of taking natural extension and Maharam extension commute:

Of course, we have $\bigvee_{n\in\mathbb{Z}} \widetilde{T}_\alpha^{-n}\mathcal{G} \otimes \mathcal{B}_+ \subset \mathcal{F} \otimes \mathcal{B}_+$. It is not difficult to check that $\bigvee_{n\in\mathbb{Z}} \widetilde{T}_\alpha^{-n}\mathcal{G} \otimes \mathcal{B}_+$ corresponds to a Maharam system that comes from a σ-algebra $\mathcal{H} \subset \mathcal{F}$. But we also have $\mathcal{G} \subset \mathcal{H}$ and as $T^{-1}\mathcal{H} = \mathcal{H}$, we get $\bigvee_{n\in\mathbb{Z}} T^{-n}\mathcal{G} \subset \mathcal{H}$. By assumption, $\bigvee_{n\in\mathbb{Z}} T^{-n}\mathcal{G} = \mathcal{F}$ and we deduce $\mathcal{H} = \mathcal{F}$ which implies $\bigvee_{n\in\mathbb{Z}} \widetilde{T}_\alpha^{-n}\mathcal{G} \otimes \mathcal{B}_+ = \mathcal{F} \otimes \mathcal{B}_+$. □

As before we deduce the following result for α-stable stationary processes:

Theorem 15. *Let $\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}, \mathbb{P}, S\right)$ be a stationary stable process with associated system $(\Omega, \mathcal{F}, \mu, T)$. If $(\Omega, \mathcal{F}, \mu, T)$ is K and not of type II_1, then $\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}, \mathbb{P}, S\right)$ is K.*

Proof. From Proposition 14, we know that the Lévy measure system of the stable process is remotely infinite. The corresponding Poisson suspension is K by a result from [14]. By applying Maruyama's representation Theorem (Theorem 6), we recover the stable process as a factor of the suspension, which therefore inherits the K property. □

Recall that in the probability preserving context, K is strictly stronger than mixing. In [11], to produce examples of mixing α-stable stationary processes that were not based on dissipative non-singular systems, the authors considered indeed null recurrent Markov chains as base systems. These systems are well known examples of K-systems, therefore Theorem 15 shows that the associated α-stable stationary processes are not just merely mixing but are indeed K.

7.3. **Rigidity.** We recall that a system $(\Omega, \mathcal{F}, \mu, T)$ is rigid if there exists an increasing sequence n_k such that $T^{n_k} \to \mathrm{Id}$ in the group of non-singular automorphism on $(\Omega, \mathcal{F}, \mu)$(the convergence being equivalent to the weak convergence in $L^2(\mu)$ of the associated unitary operators $U_{T^{n_k}} : f \mapsto \sqrt{\frac{\mathrm{d}T_*^{-n_k}\mu}{\mathrm{d}\mu}} f \circ T^{n_k}$ to the identity). Observe that in the finite measure case, rigidity doesn't imply ergodicity but prevents mixing.

Proposition 16. *The Maharam system $\left(\Omega \times \mathbb{R}_+^*, \mathcal{F} \otimes \mathcal{B}_+, \mu \otimes \frac{1}{s^{1+\alpha}} ds, \widetilde{T}_\alpha\right)$ is rigid for the sequence n_k if and only $(\Omega, \mathcal{F}, \mu, T)$ is rigid for the sequence n_k.*

Proof. First observe that the map $T \mapsto \widetilde{T}_\alpha$ is a continuous group homomorphism from the group of non-singular automorphism of $(\Omega, \mathcal{F}, \mu)$ to the group of measure

preserving automorphism of $\left(\Omega\times\mathbb{R}_+^*,\mathcal{F}\otimes\mathcal{B}_+,\mu\otimes\frac{1}{s^{1+\alpha}}\mathrm{d}s\right)$. As $T^{n_k}\to\mathrm{Id}$ then $\widetilde{T}_\alpha^{n_k}\to\mathrm{Id}$ therefore $\widetilde{T}_\alpha^{n_k}$ is rigid for the sequence n_k.

Conversely, if $\widetilde{T}_\alpha$ is rigid for the same sequence, then, as

$$\left\langle U_{\widetilde{T}_\alpha}^{n_k} f\otimes g, f\otimes g\right\rangle_{L^2\left(\mu\otimes\frac{1}{s^{1+\alpha}}\mathrm{d}s\right)}\le\|g\|_2^2\left\langle U_T^{n_k}f,f\right\rangle_{L^2(\mu)}\le\|g\|_2^2\|f\|_2^2$$

and $\left\langle U_{\widetilde{T}_\alpha}^{n_k} f\otimes g, f\otimes g\right\rangle_{L^2\left(\mu\otimes\frac{1}{s^{1+\alpha}}\mathrm{d}s\right)}\to\|g\|_2^2\|f\|_2^2$, we get $\left\langle U_T^{n_k}f,f\right\rangle_{L^2(\mu)}\to\|f\|_2^2$ thus T is rigid. □

We need the following general result:

Proposition 17. *A stationary IDp stationary process $\left(\mathbb{R}^{\mathbb{Z}},\mathcal{B}^{\otimes\mathbb{Z}},\mathbb{P},S\right)$ is rigid for the sequence n_k if and only if its Lévy measure system $\left(\mathbb{R}^{\mathbb{Z}},\mathcal{B}^{\otimes\mathbb{Z}},Q,S\right)$ is rigid for the sequence n_k.*

Proof. Consider $X:=\{X_n\}_{n\in\mathbb{Z}}$ where $X_n:=\{x_k\}_{k\in\mathbb{Z}}\mapsto x_n$ on $\mathbb{R}^{\mathbb{Z}}$ and let $\langle a,X\rangle$ be a finite linear combination of the coordinates. $\exp\mathrm{i}\langle a,X\rangle-\mathbb{E}\left[\exp\mathrm{i}\langle a,X\rangle\right]$ is a centered square integrable vector under $\mathbb{P}$ whose spectral measure (under $\mathbb{P}$) is $\lambda_a:=\left|\mathbb{E}\left[\exp\mathrm{i}\langle a,X\rangle\right]\right|^2\sum_{k=1}^{\infty}\frac{1}{k!}\sigma_a^{*k}$ where σ_a is the spectral measure of $\exp\mathrm{i}\langle a,X\rangle-1$ under Q (see [13]). Therefore $\widehat{\sigma_a}(n_k)\to\widehat{\sigma_a}(0)$ if and only if $\widehat{\lambda_a}(n_k)\to\widehat{\lambda_a}(0)$. This implies that $\exp\mathrm{i}\langle a,X\rangle-\mathbb{E}\left[\exp\mathrm{i}\langle a,X\rangle\right]$ is a rigid vector for n_k under $\mathbb{P}$ if and only if $\exp\mathrm{i}\langle a,X\rangle-1$ is a rigid vector for n_k under Q. Observe now that the smallest σ-algebra generated by vectors of the kind $\exp\mathrm{i}\langle a,X\rangle-\mathbb{E}\left[\exp\mathrm{i}\langle a,X\rangle\right]$ under $\mathbb{P}$ is $\mathcal{B}^{\otimes\mathbb{Z}}$, and the same is true with vectors of the kind $\exp\mathrm{i}\langle a,X\rangle-1$ under Q. As in any dynamical system if there exists a rigid vector for the sequence n_k there exists a non trivial factor which is rigid for the sequence n_k, we get the announced result. □

Theorem 18. *A stationary stable process $\left(\mathbb{R}^{\mathbb{Z}},\mathcal{B}^{\otimes\mathbb{Z}},\mathbb{P},S\right)$ with associated system $(\Omega,\mathcal{F},\mu,T)$ is rigid for the sequence n_k if and only $(\Omega,\mathcal{F},\mu,T)$ is rigid for the sequence n_k.*

Proof. This is the combination of the last two results. □

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